Stirling's Approximation

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I have seen three proofs of Stirling's approximation:

- the one I will outline here which idea I found in Hamming's book
- a proof based on applying the Laplace approximation to the gamma function
- probabilistic proofs relying on moment-generating/characteristic functions.

I believe the first proof is the most elementary in the sense that it requires nothing more than a first course in analysis. The idea of the proof is also very straightforward: use the trapezoidal method to approximate $\int_1^n \log x dx$ and bound the error. This said, this proof relies on two somewhat obscure facts:

Lemma 1.

$$\log \frac{1+x}{1-x} = 2\sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1}$$

whenever $|x| < 1$.

Proof. The proof is straightforward: expand $\log(1+x)$ and $\log(1-x)$ around $x = 0$ for $|x| < 1$ and subtract. $\square$

Lemma 2 (Wallis' Inequality)**.**

$$n\pi \leq \left(\frac{2^{2n}}{\binom{2n}{n}}\right)^2 \leq \pi(n+\frac{1}{2})$$

Proof. The inequality follows from the closed form of the integral

$$I_n = \int_0^{\pi/2} \sin^n x dx.$$

Clearly, $I_0 = \pi/2$ and $I_1 = 1$. Using integration by parts we can find a recurrence:

$$I_n = \int_0^{\pi/2} \underbrace{\sin^{n-1} x}_{u} \underbrace{\sin x dx}_{dv} = \left[-\cos x \sin^{n-1} x\right]_0^{\pi/2} + (n-1)\int_0^{\pi/2} \cos^2 x \sin^{n-2} x dx$$

whence we get

$$I_n = (n-1)I_{n-2} - (n-1)I_n;$$

i.e.

$$I_n = \frac{n-1}{n} I_{n-2}.$$

Using telescoping products we can find

$$I_{2n} = \frac{\pi}{2} \cdot \frac{\binom{2n}{n}}{2^{2n}} \qquad I_{2n+1} = \frac{1}{2n+1} \cdot \frac{2^{2n}}{\binom{2n}{n}}. \tag{1}$$

Now, for any $0 \le x \le 1$ the sequence $\{\sin^n x\}$ is decreasing in n, so the sequence $\{I_n\}$ must also be decreasing. Since $I_n \neq 0$ we can write

$$\frac{1}{I_{2n-1}} \le \frac{1}{I_{2n}} \le \frac{1}{I_{2n+1}} \implies \frac{1}{I_{2n-1}I_{2n}} \le \frac{1}{I_{2n}^2} \le \frac{1}{I_{2n+1}I_{2n}}.$$

Using (1) we can see

$$\frac{4n}{\pi} \le \frac{4}{\pi^2} \left(\frac{2^{2n}}{\binom{2n}{n}} \right)^2 \le \frac{4n+2}{\pi}.$$

Multiplying across by $\pi^2/4$ finishes the proof. $\square$

As mentioned, for the proof of Stirling's formula we approximately integrate $\log x$ using the trapezoidal method:

$$\int_1^n \log x dx = \sum_{i=1}^{n-1} \frac{\log(i) + \log(i+1)}{2} + E_n = \log n! - \frac{1}{2}\log n + E_n$$

where E_n is the approximation error. Noting that

$$\int_1^n \log x dx = n \log n - n,$$

we arrive at

$$n \log n - n - \log n! + \frac{1}{2} \log n = E_n;$$

i.e.

$$\log \left(\frac{(n/e)^n \sqrt{n}}{n!} \right) = E_n. \tag{2}$$

Hence, it remains to find the convergence behavior of E_n. We have

$$E_n = \sum_{i=1}^{n-1} \int_i^{i+1} \log x - \log i - (x-i) \log \frac{i+1}{i} dx = \sum_{i=1}^{n-1} \left[-1 + (i + \frac{1}{2}) \log \frac{i+1}{i} \right]$$

We can now use lemma 1 on the summand, by using the change of variables

$$u_i = \frac{1}{2i+1}$$

to get

$$\begin{aligned}
E_n &= \sum_{i=1}^{n-1} \left[-1 + \frac{1}{2u_i} \log \frac{1+u_i}{1-u_i} \right] \\
&= \sum_{i=1}^{n-1} \left[-1 + 1 + \sum_{j=1}^{\infty} \frac{u_i^{2j}}{2j+1} \right] \\
&\leq \sum_{i=1}^{n-1} \sum_{j=1}^{\infty} u_i^{2j} = \sum_{i=1}^{n-1} \frac{u_i^2}{1-u_i^2}.
\end{aligned}$$

Expanding the last term in terms of i and simplifying we get

$$E_n \leq \frac{1}{4}(1 - \frac{1}{n}).$$

At the same time, since the logarithmic function is concave, each term in the sum defining E_n is positive[1], and so E_n is monotonically increasing. Hence, E_n must be convergent to a value $E \in \mathbb{R}$. Applying this to (2) we can see that

$$\lim_{n\to\infty} \frac{n!}{e^{-E}(n/e)^n \sqrt{n}} = 1.$$

To finish the proof we need to show that $e^{-E} = \sqrt{2\pi}$. This is where we use Wallis' inequality. First, note that we can rearrange (2) to get

$$n! = e^{-E_n}(n/e)^n \sqrt{n}.$$

Substituting for $n!$ in Wallis' inequality we get

$$n\pi \leq \left(\frac{2^{2n} e^{-2E_n} (n/e)^{2n} n}{e^{-E_{2n}} (2n/e)^{2n} \sqrt{2n}} \right)^2 \leq \pi(n + \frac{1}{2});$$

i.e.

$$n\pi \leq \frac{n}{2} \left(\frac{e^{-2E_n}}{e^{-E_{2n}}} \right)^2 \leq \pi(n + \frac{1}{2}).$$

Dividing through by n and taking limits we can see that

$$\pi \leq \frac{1}{2} \left(\frac{e^{-2E}}{e^{-E}} \right)^2 \leq \pi.$$

Hence $e^{-E} = \sqrt{2\pi}$ and we arrive at

$$\lim_{n\to\infty} \frac{n!}{\sqrt{2\pi n}(n/e)^n} = 1.$$

[1]it is the area between a secant line and a positive concave function